SINR and the Poisson–Dirichlet Process

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András Tóbiás, BMS/TU Berlin, Matrikelnummer: 365856

This summary for the seminar "Zufällige Netzwerke für Kommunikation" in Summer Semester 2015 at TU Berlin extracts the paper of Keeler and Blaszczyszyn [1, pp. 2–4.], using the infinite Poisson model and its invariance of propagation in [2, pp. 1–3.] and the definition and some properties of the two-parameter Poisson–Dirichlet distribution as in [3, pp. 857–859., 861–864., 869–870., 873–874]. **In this summary we omit all proofs for brevity. The propositions the proof of which takes part in the beamer for the seminar are marked by ♯.**

1 The SINR model. Lemma 1 of [2]

We start with presenting the *infinite Poisson model* as in [2]. Here the geographic locations of the base stations form a homogeneous Poisson point process $\Phi = \{X_i\}_{i\in\mathbb{N}}$ with intensity λ on $\mathbb{R}^2$, and the 'typical user' takes place in the origin - without loss of generality, due to stationarity of $\{X_i\}$. $l(X_i)$ is the distance loss between X_i and the origin, with $l(x) = (K|x|)^\beta$ for $K, \beta > 0$.

Adding fading/shadowing to the model, the *propagation loss* is defined as $L_{X_i} = \frac{l(X_i)}{S_{X_i}}$, where $\{S_x\}_{x\in\mathbb{R}^2}$ are i.i.d. positive random variables. The power received at the origin from the base station X_i with starting power P_{X_i} is $p_{X_i} = \frac{P_{X_i}}{L_{X_i}} = \frac{P_{X_i} S_{X_i}}{l(X_i)}$. In the setting of [2] no power control is involved, hence we simply have constant power $P_{X_i} = P > 0$. Then (and also in general if the emitted powers are i.i.d.) we can formulate an equivalent model when the power is included in the associated shadowing variable: then $\tilde{S}_{X_i} = P_{X_i} S_{X_i}$ are the shadowing variables and $\tilde{P}_{X_i} = 1$ is the emitted power. This slightly simplifies the notation.

Now let $\Theta = \{Y\} = \{Y_i\}_{i\in\mathbb{N}} := \{L_{X_i}\}_{i\in\mathbb{N}}$ denote the process of propagation losses experienced in the origin with respect to the stations of Φ, as a point process in $\mathbb{R}^+$. The distribution of Θ determines all characteristics of the typical user that can be expressed in the terms of propagation losses. This motivates Lemma 1. of [2].

Lemma 1. ♯ *Assume infinite Poisson model with distance-loss l and generic shadowing variable S satisfying $\mathbb{E}(S^{\frac{2}{\beta}}) < \infty$. Then the process of propagation losses Θ experienced in the origin is a non-homogeneous Poisson point process on $\mathbb{R}^+$ with intensity measure $\Lambda([0,t]) = \mathbb{E}(\Theta([0,t])) = at^{\frac{2}{\beta}}$, where $a = \frac{\lambda\pi\mathbb{E}(S^{\frac{2}{\beta}})}{K^2}$.* [1]

We also note that the distribution of Θ is invariant with respect to the distribution of the shadowing/fading S having the same given value of the moment $\mathbb{E}(S^{\frac{2}{\beta}})$ —in particular, when we have constant shadowing $s_{const} = (\mathbb{E}(S^{\frac{2}{\beta}}))^{\frac{\beta}{2}}$. The infinite Poisson model with such a constant shadowing is equivalent to the model without shadowing ($S \equiv 1$ and the constant K is replaced by $\tilde{K} = \frac{K}{\sqrt{\mathbb{E}(S^{\frac{2}{\beta}})}}$).

We define the SINR process Ψ on $\mathbb{R}^+$ as the ratio of signal (received by the user from a certain base station) to interference (power received from all the other stations) plus noise (in our model the noise power is constant $W \geq 0$), where $Y_i = \frac{l(X_i)}{S_{X_i}}$ and $I = \sum_{Y\in\Theta} Y^{-1}$ is the power received from the whole network. We also define the signal-to-received-power-and-noise ratio (STINR) process Ψ' on $(0,1]$. Then we have

$$\Psi = \{Z\} := \left\{ \frac{Y^{-1}}{W + (I - Y^{-1})} : Y \in \Theta \right\}, \quad (1)$$

$$\Psi' = \{Z'\} = \left\{ \frac{Y^{-1}}{W + I} : Y \in \Theta \right\}.^{2} \quad (2)$$

Information on the algebraically simpler Ψ' gives information on Ψ by $Z = \frac{Z'}{1-Z'}$ and $Z' = \frac{Z}{1+Z}$.

1.1 Factorial moment measures

For $n \geq 1$, the nth *factorial moment measure* (a.k.a. nth *correlation measure*) $M'^{(n)}(t'_1, \ldots, t'_n) = M'^{(n)}((t'_1, 1] \times \ldots \times (t'_n, 1])$ of the STINR process $\{Z'\}$ is defined as

$$M'^{(n)}(t'_1, \ldots, t'_n) = \mathbb{E}\left(\sum_{\substack{(Z'_1,\ldots,Z'_n) \\ \in(\Psi')^{\times n} \\ \text{distinct}}} \prod_{j=1}^{n} \mathbf{1}_{\{Z'_j > t'_j\}} \right).$$

The equivalent measure for $\Psi = \{Z\}$ is defined analogously, but on $(t_1, \infty] \times \ldots (t_n, \infty)]$.

We need to define two integrals for both measures. $\mathcal{I}_{n,\beta}(x)$ and $\mathfrak{I}_{n,\beta}(x_1, \ldots, x_n)$ for $x, x_i \geq 0$. We will define them in the seminar. Closed forms of these integrals are in general not known, but for $n \leq 20$ the integrals are numerically tractable. Let $\hat{t}_i = \hat{t}_i(t'_1, \ldots, t'_n) = \frac{t'_i}{\sum_{j=1}^{n} t'_j}$. Then, denoting the closed unit simplex in $\mathbb{R}^n$ as Δ_n, we have the following proposition.

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[1] This lemma follows from the Displacement Theorem (Poisson Mapping Theorem), see [5, Theorem 1.3.9.].

[2] STIR: STINR without noise (i.e., with $W = 0$), SIR: SINR without noise.

Proposition 1. *For $t'_i \in [0,1]$, the factorial moment measure of order $n \geq 1$ of the STINR process Ψ' in* (1) *satisfies*[3]

$$M'^{(n)}(t'_1,\dots,t'_n) = n! \left(\prod_{i=1}^{n} \hat{t}_i^{-2/\beta}\right) \mathcal{I}_{n,\beta}(Wa^{-\frac{\beta}{2}}) \times \mathfrak{I}_{n,\beta}(\hat{t}_1,\dots,\hat{t}_n)\mathbf{1}_{\Delta_n}(t'_1,\dots,t'_n). \quad (3)$$

Moreover, for $t_i \in (0,\infty)$ the SINR process (1) has the factorial moment measure $M^{(n)}(t_1,\dots,t_n) = M'^{(n)}(t'_1,\dots,t'_n)$, where $t'_i = \frac{t_i}{1+t_i}$, $t_i = \frac{t'_i}{1-t'_i}$.

For $n \geq 1$, it is easy to see that $\mathcal{I}_{n,\beta}(0) = \frac{2^{n-1}}{\beta^{n-1}(C'(\beta))^n}$.

Proposition 2. *Let $M_0'^{(n)}$ and $M_0^{(n)}$ respectively be the factorial moment measures of the STIR and SIR processes, i.e., $M'^{(n)}$ and $M^{(n)}$ with $W = 0$. Then $M'^{(n)}(\cdot) = \frac{\mathcal{I}_{n,\beta}(Wa^{-\beta/2})}{\mathcal{I}_{n,\beta}(0)} M_0'^{(n)}(\cdot)$, $M^n(\cdot) = \frac{\mathcal{I}_{n,\beta}(Wa^{-\beta/2})}{\mathcal{I}_{n,\beta}(0)} M_0^{(n)}(\cdot)$.*

2 The PD(α,θ) process

In this section, we deal with probability distributions (V_n) with the following properties:

$$(V_n) = (V_1, V_2, \dots),\ V_1 > V_2 > \dots > 0,\ \sum_{n=1}^{\infty} V_i = 1. \quad (4)$$

Interpretation: division of a large population into a large number of possible species, and taking the limit as the number of individuals —as well as the number of species— tends to infinity. V_n represents the proportion of the nth most common species in the idealized infinite population.

Definition 1. For $0 \leq \alpha < 1$ and $\theta > -a$ suppose that the probability $\mathbb{P}_{\alpha,\theta}$ governs independent random variables U_n such that U_n has Beta$(1-\alpha, \theta + n\alpha)$ distribution[4]. Let us consider the so-called *stick-breaking model* of (U_n): $\tilde{V}_1 := U_1$,

$$\tilde{V}_n := (1-U_1)\cdots(1-U_{n-1})U_n \quad (n \geq 2), \quad (5)$$

and let $V_1 \geq V_2 \geq \dots$ be the decreasing order statistics of $\{\tilde{V}_n\}$.

Then the Poisson–Dirichlet distribution with parameters (α,θ), abbreviated as PD(α,θ) is defined as the $\mathbb{P}_{\alpha,\theta}$ distribution of (V_n).[5]

Considering (V_n) (or equivalently, $(\tilde{V}_n)$) as atoms of a point process, we can see PD(α,θ) as a distribution of a point process.

With the aim of defining the Poisson–Dirichlet process equivalently, we consider a *subordinator* $(\tau_s, s \geq 0)$, which is an almost surely increasing process with stationary independent increments and cadlag paths. We assume that (τ_s) has no drift component. Then the locations of the jumps of the subordinator $(\tau_s - \tau_{s-}, s \geq 0)$ form a Poisson point process on $[0,\infty)$. Denote the jumps of the subordinator on $(0,s)$ in decreasing order by $V_1(\tau_s) \geq V_2(\tau_s) \geq \dots$ Then, $\tau_s = \sum_{i=1}^{\infty} V_i(\tau_s)$. Thus, knowing the Poisson process of jumps we know the subordinator. Denoting the intensity measure of this Poisson point process on $(0,\infty)$ by $s\Lambda(\mathrm{d}x)$, the Laplace transform of the subordinator is

$$\mathbb{E}[\exp(-\lambda\tau_s)] = \exp\left(-s\int_0^{\infty} \exp(-\lambda x)\Lambda(\mathrm{d}x)\right).^{6} \quad (6)$$

Let $0 < \alpha < 1$. Then we call (τ_s) a stable (α) subordinator if $\Lambda = \Lambda_\alpha$, where Λ_α with $\Lambda_\alpha(x,\infty) = Cx^{-\alpha}$ $(x > 0)$, $\Lambda_\alpha(\mathrm{d}x) = Dx^{-\alpha-1}$. Then we have from (6) that $\mathbb{E}[\exp(-\lambda\tau_s)] = \exp(-sK\lambda^\alpha)$, where $K = C\Gamma(1-\alpha)$.
A crucial observation of [PY] is that for any $s > 0$ the sequence $\{\frac{V_1(\tau_s)}{\tau_s}, \frac{V_2(\tau_s}{\tau_s}, \dots\}$ has PD$(\alpha, 0)$ distribution, and also for every fixed $t > 0$, $\left(\frac{V_1(t)}{t}, \frac{V_2(t)}{t}, \dots\right)$ has PD$(\alpha,0)$ distribution[7].

2.1 STIR process is PD$(\frac{2}{\beta}, 0)$

By Lemma 1, the signals from all the base stations, or, equivalently, the *inverse values* of the propagation process Θ, form an inhomogeneous Poisson process with intensity measure $\frac{2a}{\beta}t^{-1-2/\beta}\mathrm{d}t$ (a is given in 1)[8]. Therefore, if we set $s = 1$, $\alpha = \frac{2}{\beta}$ and $C = a$, then the jumps that the stable (α) subordinator (τ_s) makes in $(0,s)$ can be identified with these signal values Y_i^{-1}. Thus, τ_1 represents the interference in our Poisson

[3] Observe *noise factorization* in the formula! The proof of this proposition uses the memory-less property of the exponential distribution and a coordinate change inspired by n-dimensional spherical coordinates. See in [4, pp. 20–23.].

[4] For $a, b > 0$, the Beta(a,b) distribution has density $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}\mathbf{1}_{[0,1]}(x)$. In an i.i.d. sample of n elements from a uniform distribution on $(0,1)$, then the kth smallest element of this sample has distribution Beta$(k, n-k+1)$.

[5] In the talk we also introduce the concept of a *size-biased permutation* of the sequence (V_n) in the setting (4), the fact that the two-parameter Poisson–Dirichlet distribution is invariant under size-biased permutation, and the interpretation of this property for the STIR process, the so-called *randomized access policy.* In fact, if $(V_n) = (V_1, V_2, \dots)$ is as in definition 1, then $(\tilde{V}_i)$ is a size-biased permutation of (V_i), see [3, pp. 856–857.].

[6] See [5, p. 6] about the Laplace functional of a Poisson point process.

[7] The proof of this observation [3, Proposition 6.] exceeds the frames of our talk.

[8] This follows by writing $\Lambda_{\Theta^{-1}}((\frac{1}{t},\infty)) = \Lambda_\Theta([0,t))$ and then differentiating this cumulative distribution function.

model, with Laplace transform $\mathbb{E}[\exp(-\lambda\tau_1)] = \mathbb{E}[\exp(-\lambda I)] = \exp[-a\Gamma(1-2/\beta)\lambda^{2/\beta}]$.

Thus, the subordinator representation of the PD process relates the PD$(\alpha, 0)$ process to our STIR process as follows. If $\{Z'_{(i)}\}$ denote the decreasing order statistics of $\{Z'\}$, then we have:

Proposition 3. *Assume $W = 0$.*
Then the sequence $\{Z'_{(i)}\}$ equals $\{V_i\}$ in distribution to for $\alpha = \frac{2}{\beta}, \theta = 0$. I.e, the STIR process Ψ' is a PD$(\frac{2}{\beta}, 0)$ *point process.*

3 Consequences of the PD-SINR correspondence

Knowing Proposition 3, we can apply many results known about the PD$(\frac{2}{\beta}, 0)$ process to the STINR, SINR and STIR processes. The first example shows that the ratios of successive STINR values have beta distributions. Proving this requires another proposition first [3, Proposition 10.], which takes part in the talk with proof.

Proposition 4. ♯ *If (V_n) has* PD$(\alpha, 0)$ *distribution ($\alpha \in (0, 1)$), then $R_n = \frac{V_{n+1}}{V_n}$ has* Beta$(n\alpha, 1)$ *distribution, i.e., $\mathbb{P}(R_n \le x) = x^{n\alpha}$ for $x \in [0, 1]$, and the R_n are mutually independent.*

Since (V_n) can be recovered from (R_n):

$$V_1 = \frac{1}{1 + R_1 + R_1R_2 + R_1R_2R_3 + \ldots}, \qquad V_{n+1} = V_1R_1 \ldots R_n \text{ for } n \ge 1, \quad (7)$$

the next corollary follows from proposition 4:

Corollary 1. *Suppose (R_n) is a sequence of independent random variables with $R_n \sim$* Beta$(n\alpha, 1)$ *for all $n \ge 1$ and for some $0 < \alpha < 1$. Then (V_n) defined by (7) has* PD$(0, \alpha)$ *distribution.*

Now, considering these propositions with $\alpha = \frac{2}{\beta}$, we conclude for the STINR process.

Proposition 5. ♯ *For the STINR process Ψ' ($W \ge 0$), $\frac{W}{I} = \frac{1}{\sum_{i=1}^{\infty} Z'_{(i)}} - 1$ and $W + I = (\frac{L}{\alpha})^{\frac{-\beta}{2}}$, where the limit $L := \lim_{i\to\infty} i(Z'_i)^{\frac{2}{\beta}}$ both exists almost surely and for all p-means with $p \ge 1$.*

Proposition 6. ♯ *For the STINR process Ψ' ($W \ge 0$), the random variables $R_i := \frac{Z'_{(i+1)}}{Z'_{(i)}}$ have, respectively,* Beta$(\frac{2i}{\beta}, 1)$ *distributions, furthermore R_i are mutually independent.*

Finally, we mention a consequence for the factorial moment measures.

Let $c_{n,\alpha,\theta} = \prod_{i=1}^{n} \frac{\Gamma(\theta+1+(i-1)\alpha)}{\Gamma((1-\alpha))\Gamma(\theta+i\alpha)}$; in particular $c_{n,2/\beta,0} = \frac{2/\beta)^{n-1}(n-1)!}{(\Gamma(2n/\beta)\Gamma(1-2/\beta)^n)}$.

Proposition 7. *The nth factorial moment density (a.k.a. the nth correlation function) of the STINR process Ψ' ($W \ge 0$) is given by*

$$\mu^{(n)}(t'_1, \ldots, t'_n) := (-1)^n \frac{\partial n M'^{(n)}(t'_1, \ldots, t'_n)}{\partial t'_1 \ldots \partial t'_n} = c_{n,\frac{2}{\beta},0} \frac{\mathcal{I}_{n,\beta}(Wa^{-\beta/2})}{\mathcal{I}_{n,\beta}(0)} \left(\prod_{i=1}^{n} t_i'^{-(\frac{2}{\beta}+1)}\right) \left(1 - \sum_{j=1}^{n} t'_j\right)^{\frac{2n}{\beta}-1} \quad (8)$$

for $(t'_1, \ldots, t'_n) \in \Delta_n$ and zero elsewhere.

This proposition applies a result about the factorial moment densities of the PD(α, θ) process [6, Theorem 2.1] to the STIR process, which is PD$(\frac{2}{\beta}, 0)$. The noise factorization in (3) makes it possible to conclude for the STINR process.[9]

References

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[9]The reason why this result of [1] is important is that nobody had been able to show the equivalence of propositions 1 and 7 before, either by differentiating the measure (3) or integrating the density (8). In the talk we point out how the corresponding result of [6] uses the fact that the PD(α, θ) distribution is invariant under size-biased permutation.